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LETTER TO THE EDITOR

Ising-like quality of the Potts model

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Abstract. A projection operator which maps the q-state Potts model onto an Ising model is investigated. The case of an approximate mapping between two-spin nearest-neighbour clusters on a d-dimensional hypercubic lattice is considered. In the ferromagnetic case, first-order transitions occur for $q > q_c(d) = 1 + \exp[2K_{Ic}(d)] \geq 2$, where $K_{Ic}(d)$ is the critical Ising coupling. Among the results are $q_c(d) \to \exp[2/(d-1)]$ as $d \to 1^+$, $q_c(2) = 3.41$ and $q_c(3) = 2.56$. The antiferromagnetic Potts model has no long range order for $q > 2$.

The Potts model is a particular generalisation of the Ising model [1-3]. Like the Ising model, its nearest-neighbour (NN) energy spectrum consistes of two levels. However, the two models differ in the degeneracy of these levels. The statistical variable at each lattice site i in the Potts model is a 'spin' σ_1 which takes on q values $\sigma_i = 1, 2, 3, \ldots q$. The q^2 states available to each NN pair are partitioned into the two energy levels with q states in one level and the remaining $q(q-1)$ states in the other energy level. This results in the reduced Potts Hamiltonian

$$\mathscr{H}_P = -H_P/k_B T = K_P \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j} \tag{1}$$

where $\delta_{\sigma_i \sigma_j}$, the Kronecker delta symbol, is unity if $\sigma_i = \sigma_j$, and zero otherwise. If the NN coupling $K_P > 0$, the model is ferromagnetic, whereas the antiferromagnetic model corresponds to $K_P < 0$. As a special case, the Ising model is recovered for $q = 2$.

Although the Potts and Ising models are similar in that both are two-level systems, the differences in the degeneracy of the levels gives rise to very different phase transition properties. For example, the Ising model ($q = 2$) possesses an ordinary critical point at $K_P = K_{Pc}(d)$ for spatial dimensionalities $d > 1$. The only known *exact* Ising solution with a critical point is the $d = 2$ case, solved by Onsager [4] with $K_{Pc}(2) = \ln(1+\sqrt{2})$. The values $K_P > K_{Pc}(d)$ at zero external field correspond to a line of first-order transitions (coexistence curve) of the Ising model.

On the other hand, the $q > 2$ Potts model displays a richer structure. For q larger than some critical value, $q_c(d)$, the Potts model has no critical point, only a first-order transition point. This was proved by Baxter [5, 6] for the $d = 2$ case where it was found that the exact value is $q_c(2) = 4$. For $q \leq q_c(d)$, the transition is second order with exponents that continuously vary with q [7-10].

The determination of $q_c(d)$ has proven to be a difficult task in general. Mean-field (MF) theory predicts that $q_c(d) = 2$ for all dimensionalities [11, 12], in contradiction with results that include fluctuations [3]. In three dimensions, series expansion methods

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indicate $q_c(3)=2.57$ [13], whereas in $d=4-\varepsilon$ dimensions, renormalisation group (RG) calculations give $q_c(4-\varepsilon)=2+\varepsilon+O(\varepsilon^2)$ [14], and for $d\geqslant 4$, $q_c(d)=2$ [14, 15]. An RG study of the mapping of the $d=2$ Potts model onto an equivalent $d=1$ quantum model yielded $q_c(2)=6.81$ [16]. On the other hand, approximate calculations involving real-space renormalisation group (RSRG) methods have obtained $q_c(d)\to\exp[2/(d-1)]$ as $d\to 1^+$ [10, 17, 18], $q_c(1.58)=12.6$ [10], $q_c(2)=3.81$ [18], 4.08 [10], 4.73 [8] and $q_c(2.32)=2.85$ [10]. The success of the RSRG methods, however, is due to enlarging the parameter space of the Potts model (1) to include 'vacany' degrees of freedom [8].

In this letter we describe an alternate method for obtaining approximate values for $q_c(d)$. We wish to emphasise that, although the formalism presented here appears entirely general, we shall only carry out an approximate calculation at the lowest order. Our method relies on exploiting the qualitative similarities of the Potts and Ising models, and in particular, the mechanism of first-order transitions in the Ising model.

To motivate our approach, recall that the MF results are asymptotically exact in the large q limit [12, 19]. Near the phase transition point, the MF free energy per site of (1), in units of k_BT, has the Landau expansion (20) in terms of the order parameter $s=(q\langle d_{\sigma_i,1}\rangle-1)/(q-1)$ [3, 21]

$$f(s)=\tfrac{1}{2}rs^2-u_3s^3+u_4s^4 \tag{2}$$

to order s^4, where the Landau coefficients are

$$\begin{aligned} r&=\frac{(q-1)}{q}(q-zK_p)\\ u_3&=\tfrac{1}{6}(q-1)(q-2)\\ u_4&=\tfrac{1}{12}(q-1)(q^2-3q+3). \end{aligned}\tag{3}$$

In (3), z is the number of NN ($z=2d$ in a d-dimensional hypercubic lattice). Note that the Ising expansion is reproduced at $q=2$. The equilibrium value(s) of the order parameter are obtained from the usual minimisation condition

$$\partial f/\partial s=0. \tag{4}$$

The mechanism for the first-order transition in the MF approximation is due to a non-zero u_3, which occurs for $q>2$ [20, 21].

Another way to view the appearance of the first-order transition is to make a shift in the order parameter

$$s=s_0+\psi \tag{5}$$

where s_0 is chosen to make the coefficient of the ψ^3 term *identically* vanish. This choice is $s_0=u_3/4u_4$. Thus (2) can be written as an expansion in ψ

$$f(\psi)=f_0-h_0\psi+\tfrac{1}{2}r_0\psi^2+u_4\psi^4 \tag{6}$$

where $f_0=f(\psi=0)$. The Landau coefficients of the shifted order parameter ψ, in terms of those in (2), are

$$\begin{aligned} h_0&=-(u_3/4u_4)[r-(u_3^2/2u_4)]\\ r_0&=r-(3u_3^2/4u_4) \end{aligned}\tag{7}$$

with u_4 unchanged due to the truncation at order s^4. The minimisation condition is

$$\partial f/\partial\psi=0. \tag{8}$$

The physical content of (2) and (6) is identical. Hence, to $O(\psi^4)$, the Landau expansion of the Potts model is simply that of an Ising model *in an external field*. It is this relationship between the Potts and Ising models that Aharony and Pytte [14] used in their RG calculation at $d=4-\varepsilon$ in the Ising-like region $q-2 \ll 1$. In a sense, their calculation takes advantage of expanding around the MF results in both d and q.

In terms of the order parameter ψ, first-order transitions occur due to the vanishing of the external field, more precisely, at $h_0=0$, $r_0<0$. This region is accessible since the transition point, $h_0=0$, translates to $r=r_1=u_3^2/2u_4>0$, and hence $r_0=-r_1/2<0$. Finally, it is easy to show that the slope of the h_0 against T curve at the transition point is negative and becomes more negative monotonically as q increases, whereas the first-order transition temperature decreases monotonically as q increases. These features are qualitatively reproduced in our method.

The result (6) shows that the Potts model can be mapped onto an equivalent Ising-like model in an external field in the MF approximation. We take this step further and propose that such a mapping can be made beyond the MF approximation for *any d and q*. This procedure is formally implemented by constructing a projection operator that maps the Potts Hamiltonian $\mathscr{H}_P$ onto an effective Hamiltonian $\mathscr{H}_I$ with Ising degrees of freedom $s_i=\pm 1$

$$\zeta \exp(\mathscr{H}_I(\{s_i\})) = \sum_{\{\sigma_i\}} \mathscr{P}([s_i\}|\{\sigma_i\}) \exp(\mathscr{H}_P(\{\sigma_i\})). \tag{9}$$

In (9), ζ is the s_i-independent part generated by the projection and $\mathscr{P}(\{s_i\}|\{\sigma_i\})$ is the projection operator from Potts to Ising degrees of freedom which must satisfy the normalisation condition

$$\sum_{\{s_i\}} \mathscr{P}(\{s_i\}|\{\sigma_i\}) = 1. \tag{10}$$

In general, $\mathscr{H}_I(\{s_i\})$ can be expected to contain not only single-site ('external field') and NN two-spin interactions, but also multi-spin interactions. Obtaining the exact solution to (9) may turn out to be as difficult as solving the original problem (1). However, we have implemented this programme in a lowest-order cluster-type approximation.

Our approximation is illustrated in figure 1. Consider a Potts two-spin NN cluster from the infinite lattice. This cluster is mapped onto an Ising two-spin NN cluster. The Ising cluster is then imbedded into the infinite lattice. All pairs are treated on an equal footing in this two-spin NN cluster approximation so the result of the mapping is an effective Ising model in an external field with NN interactions only. A similar cluster approximation was used in the study of closed-loop phase diagrams in binary fluid mixtures [22] and the re-entrant isotropic–nematic transition in biopolymers [23]. The two-spin NN cluster approximation is

$$\exp[K_{I0}+h_I(s_i+s_j)+K_I s_i s_j] = \sum_{\sigma_i,\sigma_j} \mathscr{P}_{NN}(s_i s_j \,|\, \sigma_i \sigma_j) \exp(K_P \sigma_{\sigma_i \sigma_j}) \tag{11}$$

where K_{I0} is the spin-independent constant generated by the cluster ($\zeta=\exp(zK_{I0}N/2)$, N is the total number of lattice sites), h_I and K_I are the effective Ising external field and coupling, respectively, and $\mathscr{P}_{NN}(s_i s_j \,|\, \sigma_i\sigma_j)$ is the projection operator of the two-spin NN cluster. The projection operator must not only satisfy the normalisation (10), but also have the property that $h_I \equiv 0$ at $q=2$. A projection operator which satisfies these criteria is

$$\mathscr{P}_{NN}(s_i s_j \,|\, \sigma_i\sigma_j) = \tfrac{1}{4}[1-s_i(2\delta_{\sigma_i,1}-1)(2\delta_{\sigma_i\sigma_j}-1)][1-s_j(2\delta_{\sigma_j,1}-1)(2\delta_{\sigma_i\sigma_j}-1)]. \tag{12}$$

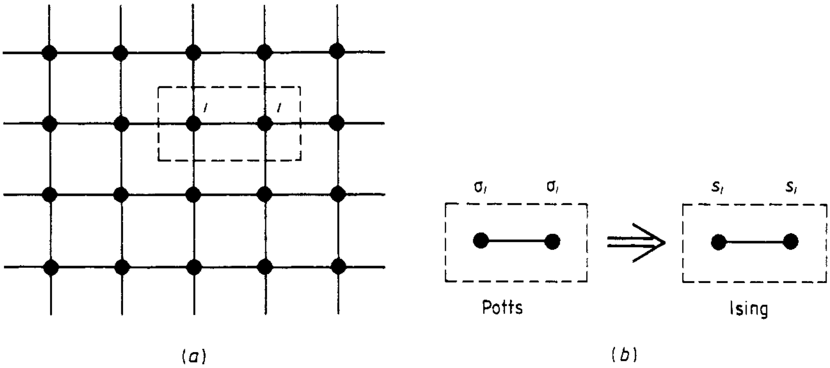

Figure 1. Illustration of the mapping of the q-state Potts model onto an effective Ising model. (a) A two-spin nearest-neighbour cluster from the d-dimensional hypercubic lattice is outlined by the broken rectangle. (b) In the approximate calculation, each Potts cluster is mapped onto an effective Ising cluster, which is then embedded into the infinite lattice.

Let us examine our choice for the projection operator (12). We focus on the ferromagnetic case $K_P > 0$. At $q = 2$, all Ising clusters are mapped onto it with equal weight. For $q > 2$, disordered Potts clusters $\sigma_i \neq \sigma_j$ contribute $2(q-1)$ states to the Ising clusters $(s_i, s_j) = (+, -)$ and $(-, +)$, and $(q-1)(q-2)$ states to $(-, -)$. The ordered Potts clusters $\sigma_i = \sigma_j$ contribute one state to $(-, -)$ and $q-1$ states to $(+, +)$. Next, consider the large q limit, $q \gg 2$, which is representative of $q > 2$. At high temperatures, $K_P \ll 1$, disordered Potts clusters dominate and the mapping is primarily to the paramagnetic cluster $(-, -)$. Hence, an Ising external magnetic field $h_I < 0$ is *dynamically generated* by the mapping. On the other hand, at low temperatures, $K_P \gg 1$, ordered Potts clusters dominate and the mapping is primarily to the paramagnetic cluster $(+, +)$. Therefore, an external field $h_I > 0$ is induced by the projection operator (12). Thus, at an intermediate temperature, $K_P = K_{P1}$, $h_I = 0$ and a first-order transition of the Potts model occurs.

Quantitative results are obtained by evaluating the statistical sum (11) for the spin configurations $(+, +)$, $(-, -)$ and $(+, -)$. This gives

$$\begin{aligned} K_{I0} &= \tfrac{1}{2}K_P + \tfrac{1}{4}\ln\{(q-1)^3[1+(q-1)(q-2)\exp(-K_P)]\} \\ h_I &= \tfrac{1}{4}\ln(q-1) - \tfrac{1}{4}\ln[1+(q-1)(q-2)\exp(-K_P)] \\ K_I &= \tfrac{1}{2}K_P - \tfrac{1}{4}\ln(q-1) + \tfrac{1}{4}\ln[1+(q-1)(q-2)\exp(-K_P)]. \end{aligned} \tag{13}$$

Note that at $q = 2$

$$\ln \zeta + \mathscr{H}_I = K_P \sum_{\langle ij \rangle} \tfrac{1}{2}(1 + s_i s_j) \tag{14}$$

as expected.

Let us study the consequences of (13) for $q > 2$. The line of first-order transitions occurs at $h_I = 0$, $K_I > K_{Ic}(d)$, where $K_{Ic}(d) > 0$ is the critical ferromagnetic Ising coupling in d dimensions. From (13), $h_I = 0$ at the value of the Potts coupling $K_P = K_{P1} = \ln(q-1)$, the first-order transition point. The value of the Ising coupling at this point is just $K_{I_i} = \frac{1}{2}\ln(q-1)$. As q decreses, K_{I_i} decreases until the first-order transition disappears at the critical point. Hence, we arrive at our main result

$$q_c(d) = 1 + \exp[2K_{Ic}(d)] \geqslant 2. \tag{15}$$

Let us consider specific cases of (15). As $d \to 1^+$, $K_{\mathrm{Ic}}(d) \to 1/(d-1)$ [24, 25], thus $q_c(d) \to \exp[2/(d-1)]$. This limiting behaviour was also found by Berker *et al* [17], Andelman and Berker [18] and Nienhuis *et al* [10] using Migdal-Kadanoff RSRG methods [24, 25] on the Potts lattice-gas model [8]. In two dimensions, the exact critical Ising coupling is $K_{\mathrm{Ic}}(2) = \frac{1}{2}\ln(1+\sqrt{2})$ [4], which gives $q_c(2) = 2+\sqrt{2} = 3.41$. In three dimensions, the series expansion result of Scesney [26] is $K_{\mathrm{Ic}}(3) = 0.222$, which

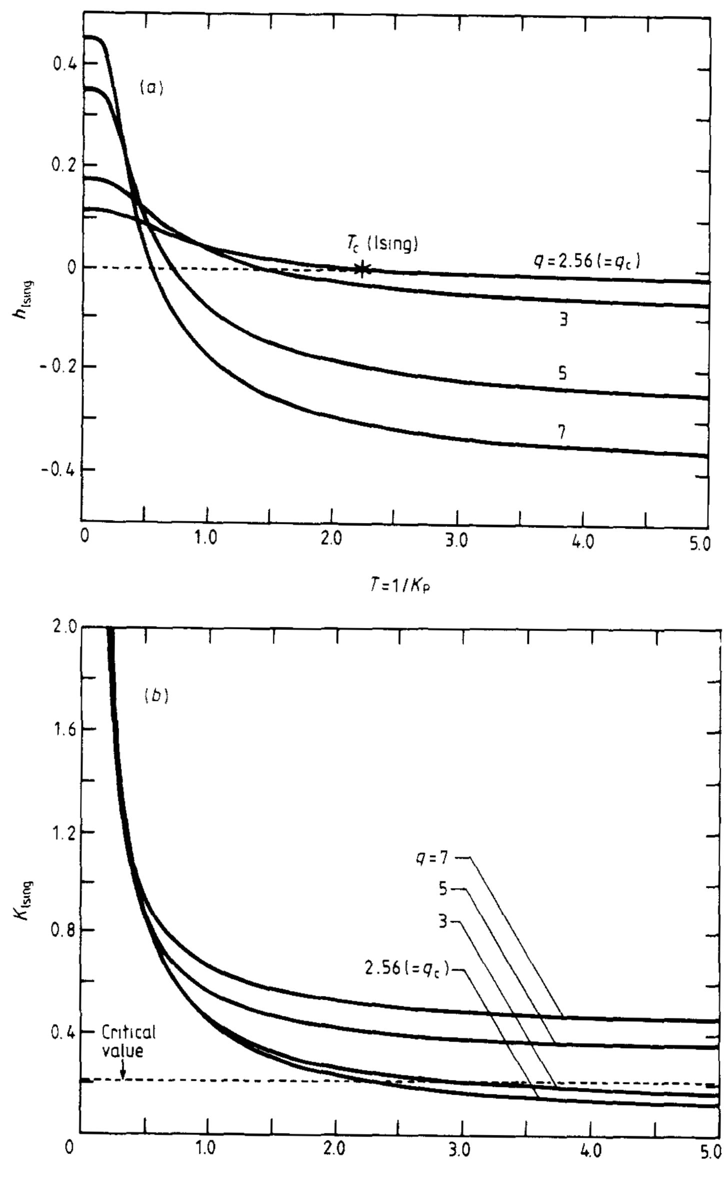

Figure 2. Plot of $h_{\mathrm{Ising}} \equiv h_I$ against $T \equiv 1/K_P$ (*a*) and $K_{\mathrm{Ising}} \equiv K_I$ against T (*b*) (cf equation (13)) for the ferromagnetic ($L_P > 0$) $d = 3$ Potts model at $q = 2.56(=q_c(3))$, 3, 5 and 7. A first-order transition occurs at $h_{\mathrm{Ising}} = 0$, $T < T_c(\mathrm{Ising})$.

gives $q_c(3) = 2.56$. This value is in agreement with the $1/q$ series expansion result of Kogut and Sinclair [13], $q_c(3) = 2.57 \pm 0.12$. The $d = 4$, 5 and 6 cases can be obtained with the aid of the results of Fisher and Gaunt [27]: $K_{Ic}(4) = 0.15$, $K_{Ic}(5) = 0.11$ and $K_{Ic}(6) = 0.09$ gives $q_c(4) = 2.35$, $q_c(5) = 2.25$ and $q_c(6) = 2.20$. Our results for $d \geq 4$, while not equal to the exact result of Aharony and Pytte [14] $q_c(d \geq 4) = 2$, asymptotically approaches their result. It is not surprising that the agreement is not quantitative

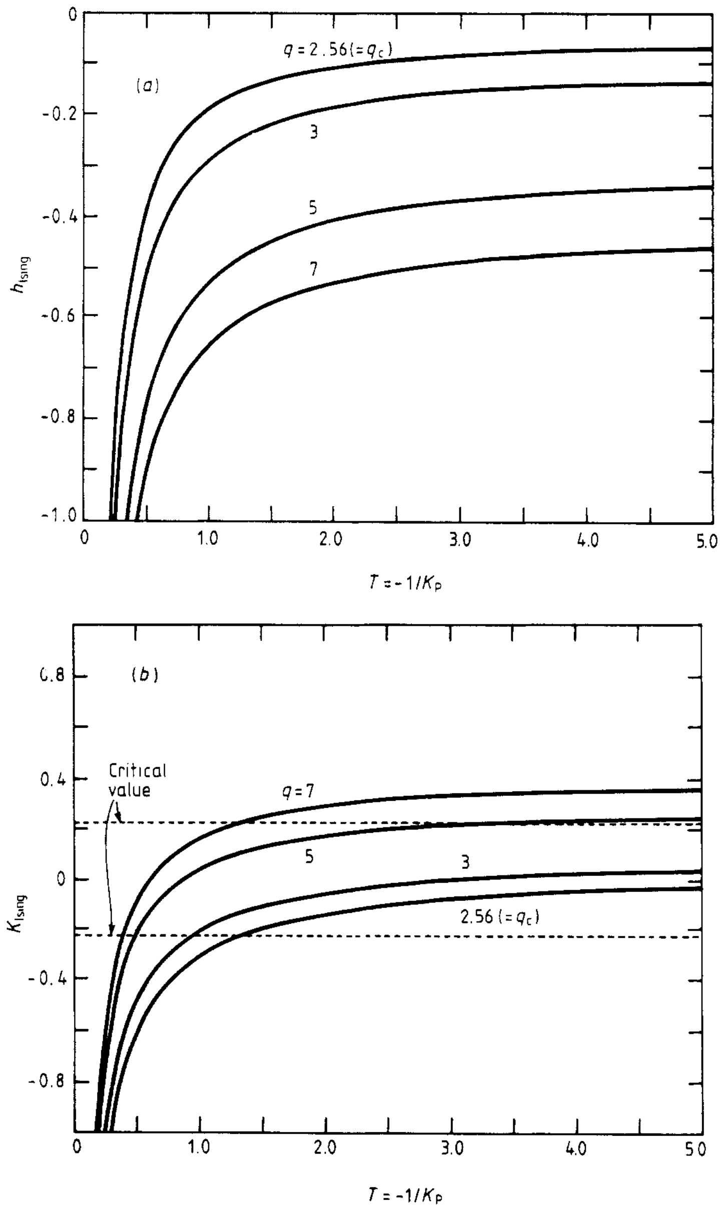

Figure 3. Plot of $h_{\text{Ising}} \equiv h_I$ against $T \equiv -1/K_P$ (a) and $K_{\text{Ising}} \equiv K_I$ against T (b) (cf equation (13)) for the antiferromagnetic ($K_P < 0$) $d = 3$ Potts model at the same values of q as shown in the ferromagnetic case, figure 2. Note that in (a), $h_{\text{Ising}} < 0$ for all T; thus there is no long range order.

for $d \geqslant 4$ since $d = 4$ is a singular point in the (d, q) plane whereas such a singularity cannot arise out of a simple finite cluster calculation. On the other hand, our results are not limited to d near four and produces the expected divergence in $q_c(d)$ as $d \to 1^+$. Also, since (15) underestimates the exact result at $d = 2$, and overestimates the exact result at $d = 4$, the $d = 3$ case may turn out to be close to the actual value since (15) must cross the exact result somewhere between two and four dimensions. This may account for the close agreement between our $q_c(3)$ result and that of Kogut and Sinclair [13]. Furthermore, our calculation is consistent with other evidence [3] that the $q = 3$, $d = 3$ Potts model is first order.

The phase diagrams of the Ising representation of the Potts model in the two-spin NN cluster approximation are shown in figures 2(*a*) and (*b*). In figure 2(*a*), note that the first-order transition temperature decreases as q increases, and that the slope of the external field h_I at the transition temperature is negative and becomes more negative as q increases, thus quantitatively following the MF results.

Finally, (13) shows that the effective Ising external field h_I is non-vanishing for the antiferromagnetic Potts model $K_P < 0$. Hence, we are led to conclude that there is no conventional long range order in the antiferromagnetic Potts model on a hypercubic d-dimensional lattice for $q > 2$ [3]. This result does not, however, preclude the existence of low temperature phases with algebraically decaying correlations [28] since the present level of approximation cannot address such phases. The phase diagrams for the antiferromagnetic case are shown in figures 3(*a*) and (*b*).

In summary, we have found a mapping of the q-state Potts model onto an effective Ising model for general q and d in a two-spin NN cluster approximation. Within this approximation, we have been able to link the critical Potts value $q_c(d)$ with the ferromagnetic Ising coupling $K_{Ic}(d)$, thus providing a unifying framework for studying the global (d, q) first-order phase diagram. We have found both qualitative and quantitative agreement with calculations based on different methods. We expect that larger clusters will not only improve the numerical accuracy of our results but provide a link to the behaviour for $q \leqslant q_c(d)$ which is beyond the region of applicability of the two-spin NN cluster approximation.

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